

Irregular shock waves formation as continuation of analytic solutions

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Dedicated to the Memory of Vladimir Shelkovich.

Abstract

This paper is devoted to the blow-up of analytic solutions with the emergence of irregular solutions. We first consider the Panov-Shelkovich system which is, as far as the author knows, the first system where such types of δ, δ' -wave solutions have been explicitly exhibited.

We propose a method to reduce this system of nonlinear PDEs to a system of two ODEs in Banach spaces, which permits to obtain theoretical existence of approximate solutions for the Cauchy problem by constructing a weak asymptotic method issued from [12] and Maslov asymptotic analysis. Further this method allows to study these PDEs through their ODEs representation by basic elementary numerical schemes very easy to construct. We then observe the expected results from the previous theoretical method; this also gives confidence in the mathematical proofs. We prove in scales of Banach spaces that this method gives back the classical analytic solutions. We also study solutions in the form of $(\delta^n)'$ for other similar systems. Indeed we show formation of very irregular shock waves when the existence time of a classical analytic solution is over.

Then we study extensions in 2-D of similar systems exhibiting irregular solutions for which this method permits to obtain existence of approximate solutions for the Cauchy problem. We finally sketch adaptations to provide a weak asymptotic method to the 3-D Euler-Poisson equations with application to pressureless fluid dynamics and cosmology [4, 5, 7].

AMS classification: 35A10, 35A24, 35D99, 35L67.

Keywords: irregular shock waves, weak asymptotic methods, Panov-Shelkovich systems, partial differential equations, ordinary differential equations in Banach spaces.

This research was supported by the Fundação de Amparo a Pesquisa do Estado de São Paulo, processo 2012/15780-9.

1. Introduction. Systems of conservation laws can admit irregular solutions that, further, can emerge from regular ones: for instance δ -shock waves have been intensively studied in recent years, [1, 2, 3, 8, 14, 18, 23, 25], among other references. δ -shock wave generation from continuous data has been studied in [8, 9, 18] by a method of new characteristics that permits to understand geometrically their formation and their interaction. The $\delta^{(n)}$ -shock waves were introduced and studied in [22, 23, 24]. Like the δ -waves these $\delta^{(n)}$ -waves are put in evidence and studied through the concept of weak asymptotic method, issued from [12], inspired by Maslov asymptotic analysis [16], which consists in exhibiting a family of approximate solutions $S(x, y, z, t, \epsilon)$ that tend to satisfy the equations in the sense of distributions in x, y, z when the parameter ϵ tends to 0 [8, 12, 13, 14, 21, 22, 23, 24, 25].

Motivated by the problem to put in evidence generation of $\delta^{(n)}$ -shock waves as time continuations of analytic solutions of the Panov-Shelkovich system, we construct a weak asymptotic method for the Cauchy problem to a family of systems containing the original Panov-Shelkovich system, that also applies for systems in nontriangular form.

In this paper we first consider the triangular system

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(au^2) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(buv) = 0, \quad (2)$$

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}(cuw) + \frac{\partial}{\partial x}(P(v)) = 0, \quad (3)$$

on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, with $a, b, c \in \mathbb{R}$, P a polynomial in one variable with real coefficients and arbitrary degree n . The periodicity assumption does not bring any restriction to the problem as long as one studies solutions in finite space and finite time intervals and as long as waves propagate with finite speed, which is the case in the present paper. With

$a = 1, b = 2, c = 2, P(v) = 2v^2$ one recovers the original Panov-Shelkovich system [22, 23, 24] for δ' -shock waves. With $P(v) = v^n, n > 2$, one obtains far more irregular types of solutions than $\delta^{(n)}$ -shock waves. They are defined as usual through the weak asymptotic method, see [10, 11, 17] for similar objects such as infinitely narrow solitons. The weak asymptotic method presented here also extends to nontriangular systems with arbitrary L^1_{loc} initial data on the multidimensional torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$.

We construct this weak asymptotic method by replacing the system (1-3) by a family of two ODEs in a classical Banach space of continuous functions. To visualize a solution it suffices to use classical numerical approximations for the solution of these ODEs: here we use the explicit Euler order one scheme in time to observe very conveniently the creation of these irregular shock waves as time continuations of analytic Cauchy-Kovalevska solutions and their behavior, in a rigorous mathematical way, from classical convergence results for ODEs.

The method presented here can be extended to weak asymptotic methods based on suitable a-priori estimates for 3-D systems of pressureless fluids without self-gravitation and with self-gravitation [7]. The method of construction of sequences for the asymptotic analysis has been inspired from the sequences constructed for the numerical schemes in [4, 5, 6] which has been a preliminary step.

2. Construction of a weak asymptotic method for system (1-3).

We assume the initial conditions u_0, v_0, w_0 are locally integrable on the torus $\mathbb{T} = \mathbb{R} / (2\pi\mathbb{Z})$. For simplification we state the proof in the case $a = b = c = 1$ and $P(v) = v^n, n \in \mathbb{N}$ since it simplifies the formulas and makes no change in the proof without any loss of generality. The approximate initial conditions $u_0^\epsilon, v_0^\epsilon, w_0^\epsilon, (\epsilon > 0, \epsilon \rightarrow 0)$, which are regularizations of u_0, v_0, w_0 , will be chosen periodic within this period. Therefore the approximate solutions $u^\epsilon, v^\epsilon, w^\epsilon$ that we will construct will be periodic as well.

Construction of the family (u^ϵ) . We start from a family (u^ϵ) of approximate solutions of (1) such that

$$\exists M_1 > 0 / |u^\epsilon(x, t)| \leq M_1 \quad \forall \epsilon, \forall x \in \mathbb{T}, \forall t \in [0, +\infty[. \quad (4)$$

For the proof of lemma 2 we need a family (u^ϵ) such that

$$\forall \beta > 0, \forall \delta > 0 \exists \text{const} / \left| \frac{\partial}{\partial x} u^\epsilon(x, t) \right| \leq \frac{\text{const}}{\epsilon^\beta} \quad \forall x \in \mathbb{T}, \forall t \in [0, \delta]. \quad (5)$$

This property can be obtained by starting from a family (u^ν) of 2π -periodic viscous solutions of (1) with viscosity coefficient ν and by reindexing this family by means of a function $\epsilon \mapsto \nu(\epsilon)$ so as to produce a growth of $\sup_{x,t \in [0,\delta]} |\frac{\partial}{\partial x} u^\epsilon(x,t)|$ as slow as wanted when $\epsilon \rightarrow 0$, and for all given δ .

Construction of the family (v^ϵ) . To obtain the approximate solutions v^ϵ of equation (2) we do as follows. We choose as initial condition a family (v_0^ϵ) such that $\|v_0^\epsilon\|_{L^1(-\pi,+\pi)} \leq \text{const}$ independent on ϵ since $\|v_0\|_{L^1(-\pi,+\pi)} < +\infty$. In a first step, for each ϵ we obtain existence and uniqueness of a solution of a linear ODE in the Banach space $(\mathcal{C}_b(\mathbb{R}), \|\cdot\|_\infty)$ of all bounded continuous (2π -periodic) functions on \mathbb{R} equipped with the sup. norm. In a second step we obtain v^ϵ as a suitable regularization by convolution of the solution of this ODE.

To this end we note as usual

$$u^+(x) = \max(0, u(x)), \quad u^-(x) = \max(0, -u(x)), \quad (6)$$

then

$$u(x) = u^+(x) - u^-(x), \quad |u(x)| = u^+(x) + u^-(x). \quad (7)$$

For $0 < \epsilon < 1$ we consider the homogeneous linear ODE

$$\frac{d}{dt} X^\epsilon(x, t) = \frac{1}{\epsilon} [X^\epsilon(x-\epsilon, t) u^{\epsilon+}(x-\epsilon, t) - X^\epsilon(x, t) |u^\epsilon(x, t)| + X^\epsilon(x+\epsilon, t) u^{\epsilon-}(x+\epsilon, t)], \quad (8)$$

$$X^\epsilon(x, 0) = v_0^\epsilon(x). \quad (9)$$

Remark. Formula (8) is an approximation of (2), i.e. $\frac{\partial}{\partial t} X + \frac{\partial}{\partial x} (Xu) = 0$. In the particular case $u^\epsilon(x, t)$, referred to as a "velocity", has a constant sign, the right-hand side of formula (8) reduces at once to the classical discretization of the x -derivative $\frac{\partial}{\partial x} (Xu)$ to the left for positive velocity u and to the right for negative velocity. This formula that replaces the x -derivative is issued from the scheme in [4, 5, 6] and therefore it can be obtained intuitively as follows considering X as a mass density and u as a velocity of matter. Consider cells of length ϵ centered respectively at the points $x, x-\epsilon, x+\epsilon$. Then we state that the quantity $\epsilon X^\epsilon(x, t+dt)$ in the cell of center x at time $t+dt$ is equal to the quantity $\epsilon X^\epsilon(x, t)$ at time t minus the quantity $(X^\epsilon |u^\epsilon|.dt)(x, t)$ escaped from this cell between t and $t+dt$ (to the left if $u^\epsilon < 0$, to the right if $u^\epsilon > 0$) plus the quantities received by this cell from the left and from the right, namely $(X^\epsilon u^{\epsilon+}.dt)(x-\epsilon, t)$, $(X^\epsilon u^{\epsilon-}.dt)(x+\epsilon, t)$ respectively, under the assumption $|u^\epsilon dt| \leq \epsilon$ so that all matter comes from

the two closest neighbor cells only. One lets dt tend to 0 for fixed ϵ to obtain the differential equation (8) instead of the numerical scheme in [4, 5, 6] in which the "infinitesimal dt " is replaced by a fixed value Δt imposing from the requirement $\|u\|_\infty \Delta t \leq \epsilon$ a bound on the velocity. This formulation of the derivative $\frac{\partial}{\partial x}(Xu)$ is more precise than a classical quotient in that it takes into account the real phenomenon of transport of the density X by the velocity u whatever the (variable) sign of u is. Further the formulation (8) permits to use as mathematical tool for fixed $\epsilon > 0$ the theory of ordinary differential equations in Banach spaces.

Since, from (4), the coefficients u_ϵ^\pm are in $\mathcal{C}_b(\mathbb{R})$, the linear ODE (8, 9) has a global unique solution X^ϵ which is a continuous function on $[0, +\infty[$ valued in the Banach space $(\mathcal{C}_b(\mathbb{R}), \|\cdot\|_\infty)$. Then one sets

$$v^\epsilon(x, t) = (X^\epsilon(\cdot, t) * \phi_{\epsilon^\alpha})(x) \quad (10)$$

where ϕ is a \mathcal{C}^∞ function on \mathbb{R} with compact support such that $\int \phi(x)dx = 1$, $\phi_{\epsilon^\alpha}(x) = \frac{1}{\epsilon^\alpha} \phi(\frac{x}{\epsilon^\alpha})$, $\alpha \in]0, 1]$ to be chosen later. Note that v^ϵ is of class \mathcal{C}^∞ in x while X^ϵ is only \mathcal{C}^0 in x .

Lemma 1.

$$\exists C > 0 / \forall \epsilon > 0 \int_{-\pi}^{+\pi} |X^\epsilon(x, t)|dx \leq \int_{-\pi}^{+\pi} |v_0^\epsilon(x)|dx \leq C. \quad (11)$$

proof. In case X would be positive this would follow at once from an integration of (8) on a period using (7). But X can have arbitrary sign therefore a different proof is needed. Developping $X^\epsilon(x, t + dt)$ from (8) according to Taylor's formula we obtain

$$X^\epsilon(x, t + dt) = \frac{dt}{\epsilon} X^\epsilon(x - \epsilon, t) u^{\epsilon+}(x - \epsilon, t) + (1 - \frac{dt}{\epsilon} |u^\epsilon(x, t)|) X^\epsilon(x, t) + \frac{dt}{\epsilon} X^\epsilon(x + \epsilon, t) u^{\epsilon-}(x + \epsilon, t) + dt \cdot o^\epsilon(x, t)(dt)$$

where $o^\epsilon(x, t)(dt) \rightarrow 0$ uniformly in (x, t) , $x \in \mathbb{R}$, $t \in I \subset \subset [0, +\infty[$, when $dt \rightarrow 0$. This uniform bound on $o^\epsilon(x, t)(dt)$ follows from the application to the \mathcal{C}^1 map $t \mapsto X^\epsilon(\cdot, t)$ of the mean value theorem under the form: if f is a \mathcal{C}^1 function then $\|f(t + dt) - f(t) - f'(t).dt\| \leq \sup_{0 < \theta < 1} \|(f'(t + \theta dt) - f'(t)).dt\|$ and from the uniform continuity of $\frac{dX^\epsilon(\cdot, t)}{dt}$ in t , taking values in the Banach space $\mathcal{C}_b(\mathbb{R})$, on any compact set in t -variable. In order to have $1 - \frac{dt}{\epsilon} |u^\epsilon(x, t)| \geq 0$, we choose $dt > 0$ small enough such that $\frac{dt M_1}{\epsilon} \leq 1$. Therefore taking absolute values,

$$|X^\epsilon(x, t + dt)| \leq \frac{dt}{\epsilon} |X^\epsilon(x - \epsilon, t)| u^{\epsilon+}(x - \epsilon, t) + (1 - \frac{dt}{\epsilon} |u^\epsilon(x, t)|) |X^\epsilon(x, t)| + \frac{dt}{\epsilon} |X^\epsilon(x + \epsilon, t)| u^{\epsilon-}(x + \epsilon, t) + dt \cdot |o^\epsilon(x, t)(dt)|.$$

After changes in x -variable and use of periodicity for simplification of integrals

$$\begin{aligned} \int_{-\pi}^{+\pi} |X^\epsilon(x, t + dt)| dx &\leq \frac{dt}{\epsilon} \int_{-\pi}^{+\pi} |X^\epsilon(x, t)| u^{\epsilon+}(x, t) dx + \int_{-\pi}^{+\pi} [1 - \frac{dt}{\epsilon} u^{\epsilon+}(x, t) - \frac{dt}{\epsilon} u^{\epsilon-}(x, t)] |X^\epsilon(x, t)| dx \\ &+ \frac{dt}{\epsilon} \int_{-\pi}^{+\pi} |X^\epsilon(x, t)| u^{\epsilon-}(x, t) dx + dt \int_{-\pi}^{+\pi} |o^\epsilon(x, t)(dt)| dx = \int_{-\pi}^{+\pi} |X^\epsilon(x, t)| dx + dt \cdot o_1^\epsilon(t)(dt) \end{aligned}$$

where $o_1^\epsilon(t)(dt) \rightarrow 0$ when $dt \rightarrow 0$, uniformly when t ranges in a bounded set of $[0, +\infty[$.

We divide the interval $[t, t + \tau]$ into subintervals $[t + i\frac{\tau}{n}, t + (i + 1)\frac{\tau}{n}]$, $0 \leq i \leq n - 1$, n as large as needed, and apply the above bound with $dt = \frac{\tau}{n}$ in each subinterval, i.e.

$$\int_{-\pi}^{+\pi} |X^\epsilon(x, t + (i + 1)\frac{\tau}{n})| dx \leq \int_{-\pi}^{+\pi} |X^\epsilon(x, t + i\frac{\tau}{n})| dx + \frac{\tau}{n} o_2^\epsilon(\frac{\tau}{n})$$

with o_2^ϵ independent on t from the uniformness o_1^ϵ in t : $o_2^\epsilon(dt) = \sup_t o_1^\epsilon(t)(dt)$. Adding on all i one obtains

$$\int_{-\pi}^{+\pi} |X^\epsilon(x, t + \tau)| dx \leq \int_{-\pi}^{+\pi} |X^\epsilon(x, t)| dx + n \frac{\tau}{n} o_2^\epsilon(\frac{\tau}{n}).$$

Since $o_2^\epsilon(\frac{\tau}{n}) \rightarrow 0$ when $n \rightarrow +\infty$ we have

$$\int_{-\pi}^{+\pi} |X^\epsilon(x, t + \tau)| dx \leq \int_{-\pi}^{+\pi} |X^\epsilon(x, t)| dx.$$

Finally, replacing t by 0 and τ by t one obtains

$$\int_{-\pi}^{+\pi} |X^\epsilon(x, t)| dx \leq \int_{-\pi}^{+\pi} |X^\epsilon(x, 0)| dx = \int_{-\pi}^{+\pi} |v_0^\epsilon(x)| dx \leq \text{const.}$$

□

Corollary. *There is a constant $M_2 > 0$, independent on ϵ , such that $\forall x \in \mathbb{R}, \forall t \geq 0$*

$$|v^\epsilon(x, t)| \leq \frac{M_2}{\epsilon^\alpha}, \quad \left| \frac{\partial}{\partial x} v^\epsilon(x, t) \right| \leq \frac{M_2}{\epsilon^{2\alpha}}. \quad (12)$$

proof. From (10) $|v^\epsilon(x, t)| = \left| \int X^\epsilon(x - y, t) \frac{1}{\epsilon^\alpha} \phi\left(\frac{y}{\epsilon^\alpha}\right) dy \right| \leq \int_{\text{supp } \phi} |X^\epsilon(x - y, t)| \frac{1}{\epsilon^\alpha} \|\phi\|_\infty dy \leq \text{const} \frac{1}{\epsilon^\alpha}$ from lemma 1. We obtain the second formula by a similar proof. \square

Lemma 2. *The family (u^ϵ, v^ϵ) is a weak asymptotic method for equation (2) when $\epsilon \rightarrow 0$ if $0 < \beta < \alpha$.*

proof. We have to prove that $\forall \psi \in \mathcal{C}^\infty$ on \mathbb{R} with compact support

$$I := \int \left[\frac{\partial}{\partial t} (v^\epsilon)(x, t) \psi(x) - u^\epsilon(x, t) v^\epsilon(x, t) \psi'(x) \right] dx \rightarrow 0 \quad (13)$$

when $\epsilon \rightarrow 0$. From (10)

$$I = \int \left[\left(\frac{d}{dt} X^\epsilon(., t) * \phi_{\epsilon^\alpha} \right)(x) \psi(x) - u^\epsilon(x, t) (X^\epsilon(., t) * \phi_{\epsilon^\alpha})(x) \psi'(x) \right] dx.$$

Now, from (8), since $|u| = u^+ + u^-$

$$I = \int \left\{ \frac{1}{\epsilon} [(X^\epsilon u^{\epsilon+})(x - \lambda - \epsilon, t) - (X^\epsilon u^{\epsilon+})(x - \lambda, t) - (X^\epsilon u^{\epsilon-})(x - \lambda, t) + (X^\epsilon u^{\epsilon-})(x - \lambda + \epsilon, t)] \frac{1}{\epsilon^\alpha} \phi\left(\frac{\lambda}{\epsilon^\alpha}\right) \psi(x) - u^\epsilon(x, t) X^\epsilon(x - \lambda, t) \frac{1}{\epsilon^\alpha} \phi\left(\frac{\lambda}{\epsilon^\alpha}\right) \psi'(x) \right\} dx d\lambda.$$

Changes in x -variable give

$$I = \int \left\{ \frac{1}{\epsilon} [(X^\epsilon u^{\epsilon+})(x - \lambda, t) (\psi(x + \epsilon) - \psi(x)) - (X^\epsilon u^{\epsilon-})(x - \lambda, t) (\psi(x) - \psi(x - \epsilon))] - u^\epsilon(x, t) X^\epsilon(x - \lambda, t) \psi'(x) \right\} \frac{1}{\epsilon^\alpha} \phi\left(\frac{\lambda}{\epsilon^\alpha}\right) dx d\lambda.$$

Let K be a finite interval in \mathbb{R} containing support of ψ and its translation by $\pm\epsilon$. We use that

$$\int (X^\epsilon u^{\epsilon+})(x - \lambda, t) \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon} dx = \int_K (X^\epsilon u^{\epsilon+})(x - \lambda, t) (\psi'(x) + O_{[x]}(\epsilon)) dx$$

where the notation $O_{[x]}(\epsilon)$ means a dependence on x that disappears in the next bound. From lemma 1 and from (4)

$$\int_K (|X^\epsilon u^{\epsilon+})(x - \lambda, t)| O_{[x]}(\epsilon) dx \leq M_1 |O(\epsilon)| \text{const.}$$

Therefore, with another $O(\epsilon)$,

$$I = \int [(X^\epsilon u^{\epsilon+})(x - \lambda, t) \psi'(x) - (X^\epsilon u^{\epsilon-})(x - \lambda, t) \psi'(x) - u^\epsilon(x, t) X^\epsilon(x - \lambda, t) \psi'(x)] \frac{1}{\epsilon^\alpha} \phi\left(\frac{\lambda}{\epsilon^\alpha}\right) dx d\lambda + O(\epsilon).$$

Since $u^{\epsilon+} - u^{\epsilon-} = u^\epsilon$, and after a change of variable

$$I = \int X^\epsilon(x - \epsilon^\alpha \mu, t) [u^\epsilon(x - \epsilon^\alpha \mu, t) - u^\epsilon(x, t)] \psi'(x) \phi(\mu) dx d\mu + O(\epsilon).$$

From (5), application of the mean value theorem in u^ϵ gives $|u^\epsilon(x - \epsilon^\alpha \mu, t) - u^\epsilon(x, t)| \leq \frac{\text{const}}{\epsilon^\beta} \epsilon^\alpha |\mu|$. Then from lemma 1,

$$|I| \leq \|X^\epsilon\|_{L^1(K)} \frac{\text{const}}{\epsilon^\beta} \epsilon^\alpha \int |\mu \phi(\mu)| d\mu + O(\epsilon) \leq \text{const} \cdot \epsilon^{\alpha-\beta} + O(\epsilon). \quad \square$$

Construction of the family (w^ϵ) . Now we proceed to the construction of $(w^\epsilon)_\epsilon$. We define w^ϵ as the solution of the following linear ODE with second member in the Banach space $\mathcal{C}_b(\mathbb{R})$

$$\frac{d}{dt} w^\epsilon(x, t) = \frac{1}{\epsilon} [(w^\epsilon u^{\epsilon+})(x - \epsilon, t) - (w^\epsilon |u^\epsilon|)(x, t) + (w^\epsilon u^{\epsilon-})(x + \epsilon, t)] - n(v^\epsilon)^{(n-1)}(x, t) \frac{\partial}{\partial x} v^\epsilon(x, t), \quad (14)$$

(notice that from (10) v^ϵ is a \mathcal{C}^∞ function so that the x -derivative above makes sense), with initial condition

$$w^\epsilon(x, 0) = w_0^\epsilon(x), \quad (15)$$

where $w_0^\epsilon \in \mathcal{C}^\infty(\mathbb{T})$, $\|w_0^\epsilon - w_0\|_{L^1(\mathbb{T})} \rightarrow 0$. Since u^ϵ and v^ϵ are known, the equation (14) with unknown w^ϵ is linear with second member and it has bounded coefficients u^ϵ . Therefore, for each $\epsilon > 0$ it admits a unique global \mathcal{C}^1 solution $t \mapsto w^\epsilon(t)$ valued in the Banach space $\mathcal{C}_b(\mathbb{R})$.

Lemma 3.

$$\exists C > 0 \quad / \quad \forall \delta > 0, \quad \forall t \in [0, \delta], \quad \forall \epsilon > 0, \quad \int_{-\pi}^{\pi} |w^\epsilon(x, t)| dx \leq \frac{C}{\epsilon^{(n+1)\alpha}}. \quad (16)$$

proof. If $dt > 0$ is small enough it follows from (14) that

$$\begin{aligned} w^\epsilon(x, t + dt) &= w^\epsilon(x, t) + \frac{dt}{\epsilon} [(w^\epsilon u^{\epsilon+})(x - \epsilon, t) - (w^\epsilon |u^\epsilon|)(x, t) + \\ &\quad (w^\epsilon u^{\epsilon-})(x + \epsilon, t)] - ndt(v^\epsilon)^{(n-1)}(x, t) \frac{\partial}{\partial x} v^\epsilon(x, t) + dt \cdot o^\epsilon(x, t)(dt), \end{aligned} \quad (17)$$

where $o^\epsilon(x, t)(dt)$ tends to 0 when $dt \rightarrow 0$ uniformly if $x \in \mathbb{R}$ and t in a compact set in $[0, +\infty[$ (the proof is the same as the one in lemma 1).

Therefore, since for $dt > 0$ small enough depending on ϵ , one has from (4) $1 - \frac{dt}{\epsilon}|u^\epsilon(x, t)| \geq 0 \forall x \in \mathbb{R}, \forall t \geq 0$. It follows that

$$\int_{-\pi}^{+\pi} |w^\epsilon(x, t+dt)|dx \leq \int_{-\pi}^{+\pi} \frac{dt}{\epsilon} (|w^\epsilon|u^{\epsilon+})(x-\epsilon, t)dx + \int_{-\pi}^{+\pi} (1 - \frac{dt}{\epsilon}|u^\epsilon|(x, t))|w^\epsilon(x, t)|dx + \int_{-\pi}^{+\pi} \frac{dt}{\epsilon} (|w^\epsilon|u^{\epsilon-})(x+\epsilon, t)dx + ndt \int_{-\pi}^{+\pi} |(v^\epsilon)^{n-1}| \cdot |\frac{\partial}{\partial x} v^\epsilon|(x, t)dx + dt.o^\epsilon(dt)$$

for $dt > 0$ small enough depending on ϵ . Here $o^\epsilon(dt) \leq 2\pi \sup_{x,t} |o^\epsilon(x, t)(dt)|$ tends to 0 uniformly ($t \in I \subset \subset [0, +\infty)$) when $dt \rightarrow 0$. Changes in variable, $|u^\epsilon| = u^{\epsilon+} + u^{\epsilon-}$, and the bounds (12) give

$$\begin{aligned} & \int_{-\pi}^{+\pi} |w^\epsilon(x, t+dt)|dx \leq \\ & \frac{dt}{\epsilon} \int_{-\pi+\epsilon}^{+\pi+\epsilon} (|w^\epsilon|u^{\epsilon+})(x, t)dx + \int_{-\pi}^{+\pi} |w^\epsilon(x, t)|dx - \frac{dt}{\epsilon} \int_{-\pi}^{+\pi} |w^\epsilon(x, t)|u^{\epsilon+}(x, t)dx \\ & - \frac{dt}{\epsilon} \int_{-\pi}^{+\pi} |w^\epsilon(x, t)|u^{\epsilon-}(x, t)dx + \frac{dt}{\epsilon} \int_{-\pi-\epsilon}^{+\pi-\epsilon} (|w^\epsilon|u^{\epsilon-})(x, t)dx + dt \frac{const}{\epsilon^{(n+1)\alpha}} + dt.o^\epsilon(dt). \end{aligned}$$

The periodicity of initial conditions, coefficients and second member implies periodicity of the solutions and therefore simplifications. One obtains

$$\int_{-\pi}^{+\pi} |w^\epsilon(x, t+dt)|dx \leq \int_{-\pi}^{+\pi} |w^\epsilon(x, t)|dx + dt \frac{const}{\epsilon^{(n+1)\alpha}} + dt.o^\epsilon(dt).$$

Sharing the interval $[t, t+\tau], \tau > 0$, into small subintervals $[t + i\frac{\tau}{m}, t + (i+1)\frac{\tau}{m}]$, $0 \leq i \leq m-1$, applying the above inequality with $dt = \frac{\tau}{m}$, adding on i and letting $m \rightarrow \infty$ (as in lemma 1) one obtains that

$$\int_{-\pi}^{+\pi} |w^\epsilon(x, t+\tau)|dx \leq \int_{-\pi}^{+\pi} |w^\epsilon(x, t)|dx + \tau \frac{const}{\epsilon^{(n+1)\alpha}}.$$

Finally setting $t = 0$ and $\tau = t$ one obtains (16). \square

In conclusion of this construction let us recall the assumptions done throughout it.

- First, all Cauchy data u_0, v_0, w_0 are periodic and their regularizations $u_0^\epsilon, v_0^\epsilon, w_0^\epsilon$ are chosen also periodic with same period, i.e. we consider the problem on the one dimensional torus.

- As usual u_0 is L^∞ and one considers approximate solutions u^ϵ of equation (1) satisfying (5): a reindexation in ϵ of the classical viscous solutions.

- The initial conditions v_0, w_0 are L^1_{loc} , and their regularizations $v_0^\epsilon, w_0^\epsilon$ are also chosen L^1_{loc} uniformly in ϵ .

Then we are going to prove:

Theorem 1. **Provided $0 < \alpha < \frac{1}{n+1}$ and $0 < \beta < \alpha$ the family $(u^\epsilon, v^\epsilon, w^\epsilon)_\epsilon$ is a weak asymptotic method for system (1, 2, 3).**

Proof. From the choice of classical approximate solutions for equation (1) and from lemma 2 for equation (2) it remains to prove that $\forall \psi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support

$$J := \int \left[\frac{\partial}{\partial t} (w^\epsilon) \psi - u^\epsilon w^\epsilon \psi' - (v^\epsilon)^n \psi' \right] dx dt \rightarrow 0$$

when $\epsilon \rightarrow 0$. From (14) and $|u| = u^+ + u^-$,

$$J = \int \left\{ \left[\frac{1}{\epsilon} (w^\epsilon u^{\epsilon+})(x - \epsilon, t) - \frac{1}{\epsilon} (w^\epsilon (u^{\epsilon+} + u^{\epsilon-}))(x, t) + \frac{1}{\epsilon} (w^\epsilon u^{\epsilon-})(x + \epsilon, t) - \frac{\partial}{\partial x} ((v^\epsilon)^n)(x, t) \right] \psi(x) - (u^\epsilon w^\epsilon)(x, t) \psi'(x) - (v^\epsilon)^n(x, t) \psi'(x) \right\} dx dt.$$

The two terms involving $(v^\epsilon)^n$ simplify each other. After changes in x -variable

$$J = \int \left\{ \left[\frac{1}{\epsilon} (w^\epsilon u^{\epsilon+})(x, t) \psi(x + \epsilon) - \frac{1}{\epsilon} (w^\epsilon u^{\epsilon+})(x, t) \psi(x) - \frac{1}{\epsilon} (w^\epsilon u^{\epsilon-})(x, t) \psi(x) + \frac{1}{\epsilon} (w^\epsilon u^{\epsilon-})(x, t) \psi(x - \epsilon) - (u^\epsilon w^\epsilon)(x, t) \psi'(x) \right] \right\} dx dt.$$

If I is a finite interval containing the support of ψ and the translated of this support by $\pm\epsilon$ one has

$$\int_I (w^\epsilon u^{\epsilon\pm})(x, t) \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} dx = \int_I (w^\epsilon u^{\epsilon\pm})(x, t) \psi'(x) dx + \int_I w^\epsilon u^{\epsilon\pm}(x, t) O_{[x]}(\epsilon) dx$$

and, from (4, 16)

$$|\int_I (w^\epsilon u^{\epsilon\pm})(x, t) O_{[x]}(\epsilon) dx| \leq M_1 \frac{const}{\epsilon^{(n+1)\alpha}} \epsilon = const. \epsilon^{(1-(n+1)\alpha)}.$$

We come back to J . Using the above and $u^\epsilon = u^{\epsilon+} - u^{\epsilon-}$, the terms involving ψ' disappear. Choosing $\alpha < \frac{1}{n+1}$ and $\beta < \alpha$ to apply lemma 2, the quantity J tends to 0 when $\epsilon \rightarrow 0$. \square

3. Numerical confirmations. The theoretical weak asymptotic method introduced in this paper permits to reduce the study of approximate solutions to the Cauchy problem for systems of PDEs such as (1-3) to a system of ODEs. Since this process is constructive one can approximate numerically

the solutions. We observed the results expected from the above proofs. This permits to visualize the δ' -waves and brings a confirmation of the theoretical results.

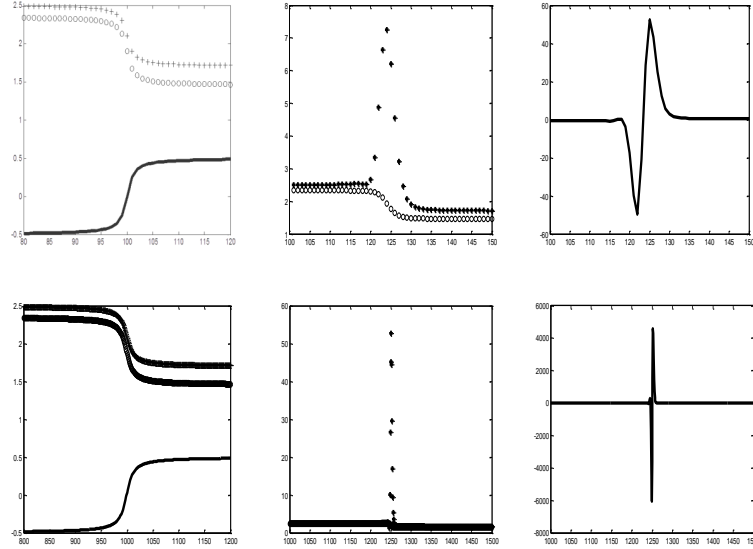


figure1. Emergence of a δ' shock wave from an analytic solution.

In the sequel we will visualize δ'' -waves from a corresponding Panov-Shelkovich system, and more general objects such as derivatives of powers of the Dirac measure. This can be done very easily from basic elementary numerical methods for ODEs. First, let us notice that the linearity of the ODEs (8, 14) implies the convergence of the explicit Euler order one method for fixed ϵ . Therefore there is no lack of rigor in using the numerical method to approximate the ODEs.

In the tests below it has sufficed to use the explicit order one Euler method, and to reproduce convolution (10) by an averaging. In numerical calculations, property (5) follows from a similar averaging as the one in [5, 6]. The scheme so obtained is very close to the scheme used in [4, 5, 6] for systems of fluid dynamics. The novelty here is that one observes the emergence of δ' -shock waves in w as a time continuation of the Cauchy-Kovalevska analytic solution when it ceases to exist.

In figure 1 one first constructs an analytic initial condition in u, v, w at time $t = 0$ (left panels). This initial condition could be a restriction to the in-

terval under concern of a set of analytic periodic functions with larger period because of the boundedness of the "velocity" u : in other words the periodic assumption causes no significative loss in generality for a study of the equations in finite space and time intervals since the velocity u is bounded. One observes numerically that this analytic solution ceases to exist at time $t = 0.037$, with the well known creation of a classical shock wave in u , as well as the emergence of a δ -wave in v and a δ' -wave in w . These last two waves grow with time and the results are given at time $t = 1$. The functions u, v, w are respectively noted with o,+ (bold + in middle panels) and continuous line. The top figures have been done with a large value of the space step so as to identify more easily the solution: it gives "thick" shock waves. The middle and right bottom figures have been done with a space step ten times smaller for a better visualization of the waves. In the middle panels we observe the usual shock wave in u and the δ -shock wave in v . In the right panels we observe the δ' -shock wave in w . Of course the aspect of the analytic initial conditions does not depend on the thickness of the space step (left panels).

Other calculations have been done by replacing v^2 in equation (3) by v^n for $n = 3, 4, 5, \dots$. One observes mathematically new highly singular shock waves in w , not pointed out by previous authors, all stemming as time continuations of analytic solutions after their analytic blow-up. These singular waves in w are located on a point; they take considerably larger values than the δ' -waves for same ϵ . This suggests that the now well known $\delta, \delta', \delta^{(n)}$ waves are only particular cases of a general phenomenon that continues the Cauchy-Kovalevska solutions after their analytic blow-up. All these one dimensional calculations are quasi instantaneous on any standard PC.

We numerically identify some of these singular shock waves for $P(v) = v^n, n = 3, 4, 5$. For convenience we consider the Riemann problem $u_l = 2, u_r = 1, v_l = 2, v_r = 1, w_l = 0, w_r = 0$ at $x = 0$ (similar results can be obtained from analytic initial conditions as in figure 1). The segment $[-0.5, 0.5]$ is divided into cells of length ϵ and we observe the solution w at time $t = 1$. The observed solution is null except on a very small region in which it has the shape of a δ' as in figure 1, but we will observe it is not a δ' if $n \neq 2$. To this end we compute a primitive of this solution : it looks like a Dirac delta measure and we compute the area of the region between the graph of the primitive and the x -axis, that should be constant (for fixed t : here $t = 1$) as a function of ϵ in case this primitive would be a Dirac delta measure. We obtain :

n	$\epsilon = 10^{-3}$	$\epsilon = 5.10^{-4}$	$\epsilon = 25.10^{-5}$	$\epsilon = 125.10^{-6}$	$\epsilon = 625.10^{-7}$	$\epsilon = 3125.10^{-8}$
2	0.084	0.087	0.089	0.090	0.090	0.090
3	0.11	0.23	0.48	0.98	1.96	4.00
4	1.36	5.56	23.2	95.7	389	1500
5	17.2	139	1168	9680	79.10^3	640.10^3

In the case $n = 2$ (i.e. Panov-Shelkovich system) the values of the areas are independent on ϵ which shows that the primitive is a Dirac delta measure, therefore $w(., t)$ is a δ' -wave. In the other cases the values of the areas depend on ϵ : for $n = 3$ they are approximately multiplied by 2 at each step to the right, and by 4, 8 if $n = 4, 5$ respectively. A power δ^α of a Dirac delta measure is represented by $(\frac{1}{\epsilon}\phi(\frac{x}{\epsilon}))^\alpha$ where ϕ is a positive continuous functions with compact support and $\int \phi(x)dx = 1$. The area between the curve and the x -axis is therefore equal to $\int \frac{1}{\epsilon^\alpha}(\phi(\frac{x}{\epsilon}))^\alpha dx = \epsilon^{1-\alpha} \int (\phi(x))^\alpha dx$. For the value $\frac{\epsilon}{2}$ the area becomes $(\frac{\epsilon}{2})^{1-\alpha} \int (\phi(x))^\alpha dx$. For $n = 3$ we numerically observe that this value is twice the value of the area relative to the space step ϵ : this gives $(\frac{\epsilon}{2})^{1-\alpha} = 2 \cdot \epsilon^{1-\alpha}$, i.e. $\alpha = 2$; for $n=4$, respectively 5, we observe that the values of the area relative to ϵ are 4, respectively 8 times the value relative to $\frac{\epsilon}{2}$. This gives $\alpha = 3$, respectively $\alpha = 4$: the observed primitives appear numerically to be powers of the Dirac delta measure. Therefore we have numerically put in evidence approximate solutions which appear to be derivatives of powers 2, 3 and 4 of the Dirac delta measure. They are obtained as continuations of the Cauchy-Kovalevska solutions (since the same results can be obtained starting from an analytic solution). In absence of definition such as the one in [15] as germs in ϵ of functions $f(x, \epsilon) = \delta(x, \epsilon)^n$, if $\delta(x, \epsilon)$ represents as usual approximations of the Dirac delta distribution, or, equivalently, asymptotic objects such as in [17], these derivatives of powers of Dirac measures are likewise to be confused with numerical blow up since they reach very high top values and are not familiar mathematical objects, since they are not defined within distribution theory. Numerically, their primitives have the aspect of Dirac delta distributions with very high peaks and an area between the peak and the x -axis that grows when the space step size ϵ diminishes, as observed in the array. The fact that these new objects are continuations of classical analytic solutions after the analytic blow up is justified by the proof in section 6 below.

4. Extension to Panov-Shelkovich $\delta^{(p)}$ -shock waves. A Panov-Shelkovich system for δ'' -shock waves can be stated, [22] p. 83,

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(u^2) = 0, \quad (18)$$

$$\frac{\partial}{\partial t}v + 2\frac{\partial}{\partial x}(uv) = 0, \quad (19)$$

$$\frac{\partial}{\partial t}w + 2\frac{\partial}{\partial x}(uw) + 2\frac{\partial}{\partial x}(v^n) = 0, \quad n = 2, \quad (20)$$

$$\frac{\partial}{\partial t}Z + 2\frac{\partial}{\partial x}(uZ) + 6\frac{\partial}{\partial x}(vw) = 0. \quad (21)$$

To obtain a weak asymptotic method it suffices to construct $u^\epsilon, v^\epsilon, w^\epsilon$ as in section 2, then define Z^ϵ as the solution of the linear equation with second member

$$\frac{d}{dt}Z^\epsilon(x, t) = \frac{2}{\epsilon}[(Z^\epsilon u^{\epsilon+})(x-\epsilon, t) - (Z^\epsilon |u^\epsilon|)(x, t) + (Z^\epsilon u^{\epsilon-})(x+\epsilon, t)] - 6\frac{\partial}{\partial x}[(v^\epsilon w^\epsilon)(\cdot, t) * \phi_{\epsilon^\gamma}](x) \quad (22)$$

for $\gamma > 0$ small enough, and with regularized initial condition Z_0^ϵ . Of course as in section 2 the numerical coefficients play no role in the proof and the result extends without any change in proof to far more general situations in one space dimension. In the proof below we set all coefficients equal one for simplification. The coefficients in (22) will be used in the numerical test depicted in figure 2. We state same assumptions on u_0, v_0, w_0 and their regularizations as above; Z_0 and its regularizations Z_0^ϵ are assumed to be L_{loc}^1 and periodic with same period. By induction the result holds clearly for $\delta^{(p)}$ Panov-Shelkovich shock-waves with arbitrary $p \in \mathbb{N}$, [22].

Theorem 2. **Provided $\alpha, \beta, \gamma > 0$ small enough and $\beta < \alpha$ the family $(u^\epsilon, v^\epsilon, w^\epsilon, Z^\epsilon)_\epsilon$ provides a weak asymptotic method for system (18-21).**

proof. We have to prove that $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$I := \int \left\{ \frac{\partial}{\partial t}(Z^\epsilon)\psi - (u^\epsilon Z^\epsilon)\psi' - (v^\epsilon w^\epsilon)\psi' \right\} dx \rightarrow 0 \quad (23)$$

when $\epsilon \rightarrow 0$. From (22), with coefficients 2 and 6 replaced by 1,

$$I = \int \left\{ \frac{1}{\epsilon}[(Z^\epsilon u^{\epsilon+})(x-\epsilon, t) - (Z^\epsilon(u^{\epsilon+} + u^{\epsilon-}))(x, t) + (Z^\epsilon u^{\epsilon-})(x+\epsilon, t)]\psi(x) + ((v^\epsilon w^\epsilon) * \phi_{\epsilon^\gamma})(x, t)\psi'(x) - (u^\epsilon Z^\epsilon)(x, t)\psi'(x) - (v^\epsilon w^\epsilon)(x, t)\psi'(x) \right\} dx.$$

One can share I into $I = I_1 + I_2$ with

$$I_1 = \int \left\{ (Z^\epsilon u^{\epsilon+})(x, t) \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} - (Z^\epsilon u^{\epsilon-})(x, t) \frac{\psi(x) - \psi(x-\epsilon)}{\epsilon} - (u^\epsilon Z^\epsilon)(x, t)\psi'(x) \right\} dx,$$

$$I_2 = \int \{(v^\epsilon w^\epsilon)(x-y, t) \frac{1}{\epsilon^\gamma} \phi(\frac{y}{\epsilon^\gamma}) - (v^\epsilon w^\epsilon)(x, t) \frac{1}{\epsilon^\gamma} \phi(\frac{y}{\epsilon^\gamma})\} \psi'(x) dx dy.$$

We first consider I_2 . After two standard changes of variables

$$I_2 = \int \{(v^\epsilon w^\epsilon)(x, t) \phi(\mu) (\psi'(x + \epsilon^\gamma \mu) - \psi'(x))\} dx d\mu.$$

Therefore, from (12) and (16),

$$|I_2| \leq \text{const.} \frac{1}{\epsilon^\alpha} \cdot \epsilon^\gamma \int_{\text{compact}} |w^\epsilon(x, t)| dx \leq \text{const} \frac{1}{\epsilon^\alpha} \cdot \epsilon^\gamma \frac{1}{\epsilon^{(n+1)\alpha}}$$

i.e.

$$I_2 \leq \text{const.} \epsilon^{\gamma - (n+2)\alpha} \quad (24)$$

which tends to 0 when $\epsilon \rightarrow 0$ provided $\gamma > (n+2)\alpha$. Now let us consider I_1 .

$$I_1 = \int \{(Z^\epsilon u^{\epsilon+})(x, t) \psi'(x) + (Z^\epsilon u^{\epsilon+})(x, t) O_{1,[x]}(\epsilon) - (Z^\epsilon u^{\epsilon-})(x, t) \psi'(x) + (Z^\epsilon u^{\epsilon-})(x, t) O_{2,[x]}(\epsilon) - (Z^\epsilon u^\epsilon)(x, t) \psi'(x)\} dx.$$

After simplification, from (4, 7)

$$|I_1| \leq \epsilon \cdot \text{const.} \int |Z^\epsilon(x, t)| dx. \quad (25)$$

Now we need to evaluate $\int |Z^\epsilon(x, t)| dx$. To this end we do as in lemma 1 and lemma 3: from (22) (dropping the coefficients for simplification)

$$Z^\epsilon(x, t + dt) = Z^\epsilon(x, t) + \frac{dt}{\epsilon} [(Z^\epsilon u^{\epsilon+})(x - \epsilon, t) - (Z^\epsilon |u^\epsilon|)(x, t) + (Z^\epsilon u^{\epsilon-})(x + \epsilon, t)] - dt \int ((v^\epsilon w^\epsilon)(x - y, t) \frac{1}{\epsilon^{2\gamma}} \phi'(\frac{y}{\epsilon^\gamma}) dy + dt \cdot o^\epsilon(x, t)(dt)).$$

The proof of this formula, where $o^\epsilon(x, t)(dt)$ converges to 0 when $dt \rightarrow 0$ uniformly in $x \in \mathbb{R}$ and t in a compact set, follows from the mean value theorem as exposed in the proof of lemma 1. Taking into account simplifications in the integrals due to the periodicity, and for $dt > 0$ small enough depending on ϵ , so that $1 - \frac{dt}{\epsilon} \|u^\epsilon\|_\infty \geq 0$, one obtains as in lemmas 1, 3

$$\int_{-\pi}^{+\pi} |Z^\epsilon(x, t + dt)| dx \leq \int_{-\pi}^{+\pi} |Z^\epsilon(x, t)| dx + dt \|v^\epsilon\|_\infty \int_{\text{compact}} |w^\epsilon(x - y, t)| dy \frac{\text{const}}{\epsilon^{2\gamma}} + dt \cdot o^\epsilon(dt),$$

i.e. from (12) and (16)

$$\int_{-\pi}^{+\pi} |Z^\epsilon(x, t + dt)| dx \leq \int_{-\pi}^{+\pi} |Z^\epsilon(x, t)| dx + dt \frac{\text{const}}{\epsilon^\alpha} \frac{1}{\epsilon^{(n+1)\alpha}} \frac{1}{\epsilon^{2\gamma}} + dt \cdot o^\epsilon(dt).$$

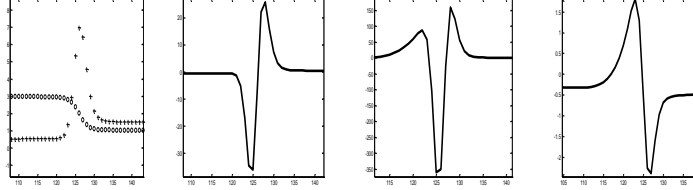


figure 2. Emergence of a δ'' -wave from an analytic solution.

Finally as in proofs of lemmas 1, 3, sharing the interval $[t, t + \tau]$ into intervals of length $\frac{\tau}{n}$ with $n \rightarrow \infty$ one obtains

$$\int_{-\pi}^{+\pi} |Z^\epsilon(x, t + \tau)| dx \leq \int_{-\pi}^{+\pi} |Z^\epsilon(x, t)| dx + \tau \frac{\text{const}}{\epsilon^{2\gamma + (n+2)\alpha}}.$$

Replacing t by 0 and τ by t one obtains that

$$\int_{-\pi}^{+\pi} |Z^\epsilon(x, t)| dx \leq \int_{-\pi}^{+\pi} |Z_0^\epsilon(x)| dx + t \frac{\text{const}}{\epsilon^{2\gamma + (n+2)\alpha}}.$$

From (25)

$$|I_1| \leq \text{const} \cdot \epsilon + \text{const} \cdot \epsilon^{1-2\gamma-(n+2)\alpha} \quad (26)$$

when t ranges in a bounded interval. From (24) and (26) $I \rightarrow 0$ when $\epsilon \rightarrow 0$ provided $\gamma > (n+2)\alpha$, $2\gamma + (n+2)\alpha < 1$ and, further, $\beta < \alpha$ to apply lemma 2. \square

Numerical confirmation. In figure 2 we start from an analytic solution (u, v, w, Z) of system (18-21) and we observe the creation of the well known shock wave in u and of a δ -wave in v (left panel), of a δ' -wave in w (left-middle panel) and of a δ'' -wave in Z (right-middle panel). If δ and δ' -waves are very clearly identified, the observation of δ'' -waves (right-middle panel) is not always very clear: one observes two small positive peaks separated by a deep negative peak because δ'' -waves are derivatives of δ' -waves whose graph is nearly vertical in its middle (left-middle panel): the very large value of the derivative there can relatively hide the two positive peaks of a δ'' -wave. So an idea to observe them better is to compute a primitive of Z (right panel) to check that it has a δ' -shape. It is also convenient to observe a primitive of a primitive of Z : one obtains an affine function with a delta peak located

on the shock wave.

5. Multi-dimensional systems and other extensions. One can extend system (1, 2, 3) to 2-D as

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(f_1(u)) + \frac{\partial}{\partial y}(f_2(u)) = 0, \quad \frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(g_1(v)) + \frac{\partial}{\partial y}(g_2(v)) = 0, \quad (27)$$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (28)$$

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial x}(P(\rho)) + \frac{\partial}{\partial y}(Q(\rho)) = 0, \quad (29)$$

where (u, v) plays the role of u in system (1, 2, 3), f_1, f_2, g_1 and g_2 are smooth functions ; ρ plays the role of v ; P and Q are polynomials. The equations (27) give a 2-D discontinuous "velocity" (u, v) , (28) is a continuity equation producing delta-waves and (29) is an equation extending (3) to 2-D. We seek a solution on the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$.

We use solutions of the scalar conservation laws (27) having the properties (4, 5) in 2-D: uniform boundedness and

$$\forall \beta > 0 \forall \delta > 0 \exists \text{const} / \left| \frac{\partial}{\partial x} u^\epsilon(x, y, t) \right| \leq \frac{\text{const}}{\epsilon^\beta}, \quad \left| \frac{\partial}{\partial y} u^\epsilon(x, y, t) \right| \leq \frac{\text{const}}{\epsilon^\beta} \quad \forall (x, y) \in \mathbb{T}^2 \forall t \in [0, \delta], \quad (30)$$

with same property for v^ϵ . For equation (28) we state the ODE as follows ([7] section 7), which consists in stating (8) in both axis directions :

$$\begin{aligned} \frac{d}{dt}X^\epsilon(x, y, t) = & \frac{1}{\epsilon} [X^\epsilon(x - \epsilon, y, t)u^{\epsilon+}(x - \epsilon, y, t) - X^\epsilon(x, y, t)|u^\epsilon(x, y, t)| \\ & + X^\epsilon(x + \epsilon, y, t)u^{\epsilon-}(x + \epsilon, y, t) + X^\epsilon(x, y - \epsilon, t)v^{\epsilon+}(x, y - \epsilon, t) - X^\epsilon(x, y, t)|v^\epsilon(x, y, t)| + \\ & X^\epsilon(x, y + \epsilon, t)v^{\epsilon-}(x, y + \epsilon, t)], \end{aligned} \quad (31)$$

$$X^\epsilon(x, y, 0) = \rho_0^\epsilon(x, y) \quad (32)$$

and

$$\rho^\epsilon(x, y, t) = (X^\epsilon(., ., t) * \phi_{\epsilon^\alpha})(x, y) \quad (33)$$

where ϕ is a \mathcal{C}^∞ function on \mathbb{R}^2 with compact support such that $\int \phi(x, y) dx dy = 1$, $\phi_{\epsilon^\alpha}(x, y) = \frac{1}{\epsilon^{2\alpha}} \phi\left(\frac{x}{\epsilon^\alpha}, \frac{y}{\epsilon^\alpha}\right)$.

We sketch the proofs which are direct extensions of the proofs in the 1-D case.

Lemma 4. $\int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} |X^\epsilon(x, y, t)| dx dy \leq \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} |\rho_0^\epsilon(x, y)| dx dy \leq \text{constant}$ independent on ϵ .

proof. Once one has written the ODE in the form

$$X^\epsilon(x, y, t+dt) = \frac{dt}{\epsilon} X^\epsilon u^{\epsilon+}(x-\epsilon, y, t) + \frac{dt}{\epsilon} X^\epsilon v^{\epsilon+}(x, y-\epsilon, t) + (1-\frac{dt}{\epsilon}) |u^\epsilon(x, y, t)| - \frac{dt}{\epsilon} |v^\epsilon(x, y, t)| X^\epsilon(x, y, t) + \frac{dt}{\epsilon} X^\epsilon u^{\epsilon-}(x+\epsilon, y, t) + \frac{dt}{\epsilon} X^\epsilon v^{\epsilon-}(x, y+\epsilon, t) + dt \cdot o^\epsilon(x, y, t)(dt), \quad (34)$$

the proof, based on periodicity, is the same as the proof of lemma 1. \square

Corollary. $|\rho^\epsilon(x, y, t)| \leq \frac{M}{\epsilon^{2\alpha}}, \quad |\frac{\partial}{\partial x} \rho^\epsilon(x, y, t)|, \quad |\frac{\partial}{\partial y} \rho^\epsilon(x, y, t)| \leq \frac{M}{\epsilon^{3\alpha}}.$

The proof follows from the convolution (33). Using the proof of lemma 2 in the 2 variables x, y one obtains that the family $(u^\epsilon, v^\epsilon, \rho^\epsilon)$ is a weak asymptotic method for equation (28).

For equation (29) we state

$$\frac{d}{dt} w^\epsilon(x, y, t) = \frac{1}{\epsilon} [(w^\epsilon u^{\epsilon+})(x-\epsilon, y, t) - (w^\epsilon |u^\epsilon|)(x, y, t) + (w^\epsilon u^{\epsilon-})(x+\epsilon, y, t) + (w^\epsilon v^{\epsilon+})(x, y-\epsilon, t) - (w^\epsilon |v^\epsilon|)(x, y, t) + (w^\epsilon v^{\epsilon-})(x, y+\epsilon, t)] - P'(\rho^\epsilon(x, y, t)) \frac{\partial}{\partial x} \rho^\epsilon(x, y, t) -$$

$$Q'(\rho^\epsilon(x, y, t)) \frac{\partial}{\partial y} \rho^\epsilon(x, y, t). \quad (35)$$

Lemma 5. $\exists N \in \mathbb{N}, \exists C > 0 / \forall \delta > 0, \forall t \in [0, \delta], \forall \epsilon > 0 \quad \int_{-\pi}^{+\pi} |w^\epsilon(x, y, t)| dx \leq \frac{C}{\epsilon^{N\alpha}}.$

It suffices to develop (35) according to Taylor's formula, as done in (34), with the two additional terms involving P and Q and to follow the proof of lemma 3. \square

Then a proof similar to that of theorem 1 shows that (35) provides a weak asymptotic method to equation (29).

The weak asymptotic method under consideration applies in arbitrary dimension on \mathbb{T}^d and \mathbb{R}^d . For simplicity and to facilitate the reading we consider a 2×2 system on the 1-D torus since multidimensional extensions

are rather easy [7]. Now we show that the weak asymptotic method extends easily to systems of the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(uf(u, v)) = 0 \quad (36)$$

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(vg(u, v)) = 0, \quad (37)$$

on the 1-D torus, with f, g analytic bounded on \mathbb{R}^2 , since this assumption will permit a very simple proof. This weak asymptotic method has been adapted to pressureless fluids in [7]. For system (36, 37) we consider the ODE

$$\frac{du^\epsilon}{dt}(x, t) = \frac{1}{\epsilon}[u^\epsilon(x-\epsilon, t)f^{\epsilon,+}(x-\epsilon, t) - u^\epsilon(x, t)|f^\epsilon(x, t)| + u^\epsilon(x+\epsilon, t)f^{\epsilon,-}(x+\epsilon, t)], \quad (38)$$

where $f^\epsilon = f(u^\epsilon, v^\epsilon)$ and similar ODE for (37). We sketch the proof: a detailed proof for the (more difficult) system of pressureless fluids is given in [7]. The initial data $u_0^\epsilon, v_0^\epsilon \in \mathcal{C}_b(\mathbb{T})$ with $\exists C > 0$ / $\|u_0^\epsilon\|_{L^1(\mathbb{T})}, \|v_0^\epsilon\|_{L^1(\mathbb{T})} < C \forall \epsilon > 0$.

For fixed $\epsilon > 0$ the system of two ODEs (38) admits a local solution $u^\epsilon, v^\epsilon : [0, \delta[\rightarrow \mathcal{C}_b(\mathbb{R}) \times \mathcal{C}_b(\mathbb{R})$. First we prove that

$$\|u^\epsilon(\cdot, t)\|_\infty \leq \|u_0^\epsilon\|_\infty \exp\left(\frac{\text{const}}{\epsilon}t\right). \quad (39)$$

Indeed, from (38) $|u^\epsilon(x, t)| \leq |u_0^\epsilon(x)| + \frac{\text{const}}{\epsilon} \int_0^t \|u^\epsilon(\cdot, s)\|_\infty ds$ since f is assumed bounded; then one applies the Gronwall formula. It follows easily from (39) that (38) has a global solution on $[0, +\infty[$, as in [7] section 4. Now we prove that

$$\int_0^{2\pi} |u^\epsilon(x, t)| dx \leq \int_0^{2\pi} |u_0^\epsilon(x)| dx. \quad (40)$$

Indeed, from (38) the proof of (40) is identical to the proof of lemma 1. It follows that the family (u^ϵ, v^ϵ) is a weak asymptotic method for system (36, 37): one has to prove that

$$\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}) \quad \int \left(\frac{\partial}{\partial t}(u^\epsilon)\psi - u^\epsilon f^\epsilon \psi' \right) dx \rightarrow 0$$

when $\epsilon \rightarrow 0$. From (38, 40) the proof is similar to that of lemma 2.

In [4, 5] a numerical scheme for the 3-D system of pressureless fluids and the 3-D Euler-Poisson system has been investigated mathematically and numerically. In the case of 3-D pressureless fluids without selfgravitation and 1-D Euler-Poisson equations it is proved there that this scheme provides a weak asymptotic method with weak derivative both in space and time: th. 3 p. 1909 in [4] with announcement p. 1911 of the result in 3-D proved in [5] p. 96-100, th. 2 p. 86 in [5] for the 1-D Euler-Poisson equations. Since the author was unaware of the concept of weak asymptotic method [12] the weak asymptotic method there is called "convergence" in [4] and "consistence" in [5].

For the 2-D and 3-D Euler-Poisson system the scheme in [5] fails to provide a weak asymptotic method since one is forced (th. 1 p. 85) to assume the boundedness of the velocity vector and of the gradient of the gravitation potential. This problem has been solved in [7] by an adaptation of the method presented in this paper. This adaptation relies on a priori estimates which have been avoided in the example (36, 37) by the assumption that f and g are bounded on \mathbb{R}^2 .

6. The weak asymptotic method in the analytic case. In section 2 we have proved that the solutions of the system of ODEs is a weak asymptotic method for the system (1-3), which has received a confirmation in section 3 from a numerical solution of the ODEs of section 2. Now we consider the classical case and we prove that the weak asymptotic method under consideration gives the classical analytic solution when it exists. Indeed the numerical scheme of section 3 gives a smooth solution; now we prove this smooth solution is the analytic solution as it should be. The space of all holomorphic maps from $\Omega \subset \mathbb{C}$ open into a Banach space E is denoted by $\mathcal{H}(\Omega, E)$. We assume holomorphy in the initial conditions and coefficient:

$v_0^\epsilon, w_0^\epsilon \in \mathcal{H}(\mathbb{T} \times \{y \in]-r, +r[\})$ and $u^\epsilon(., .) \in \mathcal{H}(\mathbb{T} \times \{y \in]-r, +r[\} \times \{t/|t| < a\})$ for some $r, a > 0$ uniformly in ϵ .

We assume the coefficients have a fixed sign away from 0:

$$\exists \eta > 0 / |u^\epsilon(x, t)| \geq \eta \quad \forall (x, t) \in \mathbb{T} \times \{t/|t| < a\},$$

so as to get rid of the lack of analyticity due to u^\pm and $|u|$ in (8, 14): in the following proof we assume they are positive, which is the case in figures 1 and 2. Then equation (8) becomes

$$\frac{\partial X^\epsilon}{\partial t}(x, t) = -\frac{1}{\epsilon} [X^\epsilon(x, t)u^\epsilon(x, t) - X^\epsilon(x - \epsilon, t)u^\epsilon(x - \epsilon, t)]. \quad (41)$$

It has previously been proved that a global solution exists in the space $\mathcal{C}_b(\mathbb{R})$. But we do not know the existence and the nature of a limit when $\epsilon \rightarrow 0$. The purpose of this section is to prove that *the approximate solutions converge to the classical analytic solution*. This is done by applying an abstract version of the Cauchy-Kovalevska theorem. Let us recall the version of this theorem presented in [26] theorem 17.2 p. 148.

Definition. A scale of Banach spaces is a family of Banach spaces $(E_s)_{s, 0 < s < s_0}$, such that $\forall s, s' \in]0, s_0], s > s' \Rightarrow E_s \subset E_{s'}$ with inclusion $i_{s,s'} : E_s \hookrightarrow E_{s'}$ such that $\|i_{s,s'}\|_{L(E_s, E_{s'})} \leq 1$ where $L(E_s, E_{s'})$ is the Banach space of all linear continuous maps from E_s into $E_{s'}$.

Theorem: Abstract Cauchy-Kovalevska. We consider the Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad u(0) = u_0 \in E_{s_0}, \quad (42)$$

where, for some $a > 0$, $\forall t \in \{|t| < a\} \subset \mathbb{C}$, $A(t) \in L(E_s, E_{s'})$ as soon as $s < s'$, and the map $A \in \mathcal{H}(\{|t| < a\}, L(E_s, E_{s'}))$. The map $f \in \mathcal{H}(\{|t| < a\}, E_{s_0})$. The main assumption is:

$$\exists M > 0 / \forall t \in \{|t| < a\}, \forall s > s' \quad \|A(t)\|_{L(E_s, E_{s'})} \leq \frac{M}{s - s'}. \quad (43)$$

Then $\exists C > 0$ depending only on M (not on f and u_0) such that $\forall s < s_0$ if $\delta := \min(a, \frac{C}{s_0 - s})$ $\exists!$ solution $u \in \mathcal{H}(\{|t| < \delta\}, E_s)$. Further the bounds of u depend only on domains and bounds of the data.

To apply this theorem we consider the scale of Banach spaces $(E_s)_{0 < s < s_0}$, for some $s_0 > 0$, defined by

$$E_s := \{f \in \mathcal{H}(\mathbb{T} \times]-s, s[, \mathbb{C}) \text{ continuous and bounded on the closure of this strip}\}, \quad (44)$$

equipped with the sup norm on the strip. The real number $s_0 > 0$ is chosen so that $\forall \epsilon, t$ the functions $(u^\epsilon(\cdot, t), v_0^\epsilon, w_0^\epsilon)_\epsilon$ are elements of the space E_{s_0} , uniformly bounded in sup norm. Then, if $s > s'$, for all t , the map $\mathcal{A}^\epsilon(t) \in L(E_s, E_{s'})$ is defined by:

$$[z \mapsto X(z)] \in E_s \mapsto [z \mapsto -\frac{1}{\epsilon}(X(z)u^\epsilon(z, t) - X(z - \epsilon)u^\epsilon(z - \epsilon, t))] \in E_{s'}. \quad (45)$$

We have $\|\mathcal{A}^\epsilon(t)\|_{L(E_s, E_{s'})} \leq \frac{\text{const}}{s - s'}$, where *const* is independent on s, s', t and ϵ . This follows at once from the mean value theorem applied to the second

member of (45) and from Cauchy's inequality for the derivative of a holomorphic function.

The abstract version of the Cauchy-Kovalevska theorem gives :
 $\exists C > 0, \ / \ \forall \epsilon, \ \forall s < s_0 \ \exists ! X^\epsilon \in \mathcal{H}(\{|t| < \frac{C}{s_0-s}\}, E_s)$ solution of (41) with initial condition v_0^ϵ , where C is independent on s and ϵ . Setting $\delta = \frac{C}{s_0-s}$, the functions $X^\epsilon : (z, t) \mapsto X^\epsilon(z, t)$ so obtained are holomorphic on $\mathbb{T} \times]-s, s[\times \{|t| < \delta\}$ with values in \mathbb{C} , bounded on compact sets uniformly in ϵ , from the uniform bounds and domains of v_0^ϵ and $u^\epsilon(., t)$ in ϵ and t . From the theory of normal families of holomorphic functions, from any sequence of the X^ϵ s one can extract a subsequence that converges uniformly on the compact sets of $\mathbb{T} \times]-s, s[\times \{|t| < \delta\}$. The limit when $\epsilon \rightarrow 0$ is analytic and unique extension to the complex domain of the classical solution of the linear Cauchy-Kovalevska problem

$$\frac{\partial X}{\partial t}(x, t) = -\frac{\partial(Xu)}{\partial x}(x, t), \quad X(x, 0) = v_0(x). \quad (46)$$

Therefore all subsequences of (X^ϵ) converge to the same limit. Therefore the whole sequence (X^ϵ) converges uniformly on compact sets of $\mathbb{T} \times \{0 < t < \delta\}$ to the solution of (46) when $\epsilon \rightarrow 0$, i.e. of (2), with $b = 1$ for simplicity. The proof applies as well to (14): for the second member of (14) one uses the uniform bounds and uniform domain on the v^ϵ s obtained from the abstract Cauchy-Kovalevska theorem applied to (8). The proof ensures the same existence time for the weak asymptotic method as when it is applied to prove existence of the classical analytic solution. It also applies in several space dimension.

The proof in this section can be reproduced practically without any change for systems such as (36, 37) and those in [7], using the nonlinear theorem due to L. Nirenberg and T. Nishida [19, 20] instead of the linear abstract Cauchy-Kovalevska theorem [26].

In conclusion we have proved that with analytic initial data (and constant sign of u) the weak asymptotic method starts by giving the classical analytic solution, which proves that the numerically observed very irregular shock-waves are continuations of the classical analytic solutions after the analytic blow up occurs, as observed numerically.

7. Conclusion. This new technique of construction of weak asymptotic methods permits to study PDEs having δ, δ', \dots -shock wave solutions, and other systems, such as the 3-D Euler-Poisson system [7] for applications in

fluid dynamics and cosmology [5, 6, 7], by transferring the problem to a family of ODEs in Banach spaces. This permits to demonstrate in a mathematically rigorous way the existence of approximate solutions for the Cauchy problem, extending previous results on the Riemann problem.

Furthermore this method permits to construct very simple numerical schemes that bring a confirmation of the theoretical results and permit a visualization of the approximate solutions. In particular it has permitted to put in evidence Panov-Shelkovich $\delta^{(n)}$ -shocks and much more irregular shock waves (from a more general family of systems of conservation laws) as continuations of the classical analytic Cauchy-Kovalevska solutions when their existence time is over.

Of course the problem of uniqueness of a “limit” of these approximate solutions remains unsolved. It has been checked numerically on standard systems of fluid dynamics that the adaptation of the method presented here to equations of fluid dynamics [7] and to the very closely related numerical method in [4, 5, 6] have always given the known solutions even on very demanding tests, such as tests of Woodward-Colella, Toro, Lax, see [6].

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